Entanglement classification of arbitrary-dimensional multipartite pure states under stochastic local operations and classical communication via the ranks of the coefficient matrices

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The entanglement classification of different types of multipartite entanglement has been one of the main tasks in quantum information theory. Recently, Li \textit{et al.} proposed the approach of entanglement classification under stochastic local operations and classical communication (SLOCC) for $n$-qubit pure states in terms of the ranks of the coefficient matrices [1]. We generalized Li’s approach to $n$-qudit pure states in the $n$-partite Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$ [2]. An arbitrary $n$-qudit pure state in $\mathcal{H}$ can be expanded as $|\psi\rangle = \sum_{i=0}^{\prod_{k=1}^{n} d_k - 1} a_i |s_1 s_2 \cdots s_n\rangle$, where $a_i$ are the coefficients and $|s_1 s_2 \cdots s_n\rangle$ are the basis states $|s_1 s_2 \cdots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle$ with $s_k \in \{0, 1, \cdots, d_k - 1\}$, $k = 1, \cdots, n$. The coefficient matrices $C_{1\cdots l,l+1\cdots n}(|\psi\rangle)$ corresponding to $|\psi\rangle$ is constructed by arranging $a_i$ in ascending lexicographical order ($0 < l < n$):

\[
C_{1\cdots l,l+1\cdots n}(|\psi\rangle) = 
\begin{pmatrix}
\begin{array}{cccc}
  a_0 \cdots 0 & 0 & \cdots & a_0 \cdots 0 \\
  a_0 \cdots d_l - 1 & 0 & \cdots & a_0 \cdots d_l - 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_0 \cdots d_l - 1 \cdots d_n - 1 & a_0 \cdots d_l - 1 \cdots d_n - 1 & \cdots & a_0 \cdots d_l - 1 \cdots d_n - 1 \\
\end{array}
\end{pmatrix}
\]

We proved that for qudit states, the ranks of the coefficient matrices are invariant under SLOCC. By taking all the ranks of the coefficient matrices into account, we give the entanglement classification procedure for arbitrary-dimensional multipartite pure states under SLOCC. As a main application, we study the entanglement classification for quantum systems in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{d_l}$ in terms of the ranks of the coefficient matrices.

