JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN VOL. 17, SUPPLEMENT B-I, 1962
PROCEEDINGS OF INTERNATIONAL CONFERENCE ON MAGNETISM AND CRYSTALLOGRAPHY, 1962, VOL. I

Approximate Calculation of Magnetostatic Energies*

WILLIAM FULLER BROWN, JR.

Department of Electrical Engineering, University of Minnesota
Minneapolis 14, Minnesota, U.S.A.

In domain theory and in micromagnetics, an important and often difficult step is the evaluation of the magnetostatic energy. In many cases a rigorous calculation, even if possible, is laborious and is not justified in view of the crudeness of the theoretical model. The method described here leads to lower and upper bounds for the magnetostatic self-energy of a given magnetization distribution $\boldsymbol{M}$. This is accomplished by maximizing $W_H=-(8\pi)^{-1}\int \boldsymbol{H}_2^2 dV-\int \boldsymbol{M}\cdot\boldsymbol{H}_2 dV$ and minimizing $W_B=+(8\pi)^{-1}\int \boldsymbol{B}_2^2 dV-\int \boldsymbol{M}\cdot\boldsymbol{B}_2 dV$ with respect to adjustable parameters in a conveniently chosen irrotational field $\boldsymbol{H}_2$ and solenoidal field $\boldsymbol{B}_2$ (dV=volume element). Then $W_H\leq W_m=-(1/2)\int \boldsymbol{M}\cdot\boldsymbol{H}_1 dV$ and $W_B\geq W_m'=-(1/2)\int \boldsymbol{M}\cdot\boldsymbol{B}_1 dV=W_m-2\pi\int \boldsymbol{M}^2 dV$, where $\boldsymbol{H}_1$ and $\boldsymbol{B}_1=\boldsymbol{H}_1+4\pi\boldsymbol{M}$ are the actual magnetizing force and flux density due to $\boldsymbol{M}$. By judicious choice of the forms of $\boldsymbol{H}_2$ and $\boldsymbol{B}_2$, one can obtain a rough estimate of W_m (or of W_m') by slight labor. Examples are given.

In theoretical calculations of magnetic microstructure, a difficult step is the calculation of the magnetostatic fields and self-energy. In rigorous nucleation-field calculations[1),2)], the potential problem must be solved rigorously; but in approximate calculations of the magnetization distribution by the Ritz method, a correspondingly approximate calculation of the field would be sufficient. And in domain theory, the Fourier series often used for the magnetostatic calculation are at times fantastically incongruous with the crudeness of the underlying model.

The method to be described permits a calculation of the magnetostatic energy with as much or as little accuracy as may be desired, and always with known limits to the error.

Suppose that we wish to calculate, approximately, the field of a given distribution of magnetization $\boldsymbol{M}$; and suppose for the present that we are interested in the $\boldsymbol{H}$ field. This is the Coulomb field intensity of the magnetic poles or charges. An obvious method of approximate calculation is to replace the given charge distribution, with unknown field $\boldsymbol{H}_1$, by one located roughly in the same regions, with known field $\boldsymbol{H}_2$. For example, we may replace a disk, whose field is hard to calculate, by a sphere, for which the calculation is easy. But now we can improve our approximation by making our substitute distribution depend on one or more parameters, and then adjusting the parameters for best fit in a least-squares sense. That is, we adjust them so as to minimize $\int(\boldsymbol{H}_2-\boldsymbol{H}_1)^2 dV$. In this expression, expand the square; note that the term containing $\boldsymbol{H}_1^2$ is independent of our parameters; and transform the cross-product term by noting that, since $\boldsymbol{H}_2$ is irrotational and $\boldsymbol{H}_1+4\pi\boldsymbol{M}$ is solenoidal, $\int \boldsymbol{H}_2\cdot(\boldsymbol{H}_1+4\pi\boldsymbol{M})dV=0$. The result is that we must minimize $-W_H$, or maximize W_H, where

$$W_H=-(1/8\pi)\int \boldsymbol{H}_2^2 dV-\int \boldsymbol{H}_2\cdot\boldsymbol{M}dV\,. \quad (1)$$

This principle may be interpreted as follows. Let $\boldsymbol{H}_2$ be a disembodied field, irrotational everywhere, whose self-energy is $-(1/8\pi)\int \boldsymbol{H}_2^2 dV$ and whose interaction energy with the magnetization is $-\int \boldsymbol{H}_2\cdot\boldsymbol{M}dV$; let W_H be the sum of these two energies. To make $\boldsymbol{H}_2$ approximate best the actual field of $\boldsymbol{M}$, *maximize* W_H under whatever constraints are imposed by the form chosen for $\boldsymbol{H}_2$. When there are no constraints, the maximization makes $\boldsymbol{H}_2=\boldsymbol{H}_1$ and $W_H=-\frac{1}{2}\int \boldsymbol{H}_1\cdot\boldsymbol{M}dV$,

* This work was assisted by a grant from the National Science Foundation.

the self-energy W_m of the poles. Thus a maximization under constraints leads to a value of W_H that in general is *smaller* than the actual value of the self-energy W_m.

This way of formulating the principle enables us to avoid solving any potential problem, even an easy one. We do not need to think about the substitute charges, only about the field $\boldsymbol{H}_2$. In choosing its form, however, we do well to remember that the points of divergence of $\boldsymbol{H}_2$ should be somewhere near the actual poles.

If we wish to calculate the $\boldsymbol{B}$ field rather than the $\boldsymbol{H}$ field, we use Amperian currents instead of poles. Exactly analogous reasoning then leads to the conclusion that we should *minimize*

$$W_B=+(1/8\pi)\int \boldsymbol{B}_2^2 dV-\int \boldsymbol{B}_2\cdot\boldsymbol{M}dV\,, \qquad (2)$$

where $\boldsymbol{B}_2$ is a solenoidal field. Minimization without constraints makes $\boldsymbol{B}_2$ the actual flux density due to $\boldsymbol{M}$ and gives W_B its minimum value $W_m'=-\frac{1}{2}\int \boldsymbol{B}_1\cdot\boldsymbol{M}dV$. By this method we get, in general, *too large* a value of W_m'. But

$$W_m-W_m'=\tfrac{1}{2}\int(\boldsymbol{B}_1-\boldsymbol{H}_1)\cdot\boldsymbol{M}dV=2\pi\int \boldsymbol{M}^2 dV\,, \qquad (3)$$

which for a given magnetization distribution is a known constant, and which for a homogeneous ferromagnet is $2\pi M_s^2 V$ (M_s=spontaneous magnetization, V=specimen volume). By adding this known term to our overestimate of W_m', we get an overestimate of W_m. We now have two estimates of the internal magnetostatic pole energy W_m, and we know that the correct value lies between them.

If we use complete sets of orthonormal functions, we can obtain as accurate a value as we please by keeping enough terms. For many purposes, however, a very crude approximation will suffice.

I will illustrate the method by applying it to an infinite plane slab, of thickness b, magnetized in domains of width a, alternately in opposite directions normal to the faces. This case, of course, can be solved rigorously with Fourier series[3]. To solve it approximately, we assume the simple $\boldsymbol{H}_2$ and $\boldsymbol{B}_2$ fields shown in Fig. 1. All curves shown are circles, and H and B are constants. The discontinuities across surfaces affect only the normal component of $\boldsymbol{H}_2$ in (a) and only the tangential component of $\boldsymbol{B}_2$ in (b); thus $\boldsymbol{H}_2$ does no curling and $\boldsymbol{B}_2$ no diverging. Roughly, $\boldsymbol{H}_2$ diverges where $\boldsymbol{M}$ converges, and $\boldsymbol{B}_2$ curls where $\boldsymbol{M}$ curls. By maximization and minimization with respect to the parameters H and B we get, for the magnetostatic self-energy per unit volume, $w_m=\pi M_s^2 u$, where

$$u_{\min}=1/\pi p,\ p=b/a\geq 1; \qquad (4)$$

$$u_{\max}=2-(4/\pi)p\ln(1+\pi/2p)\,. \qquad (5)$$

Comparison with the rigorous values shows that $u_{\min}/u\cong 0.6$ and $u_{\max}/u\cong 2.9$.

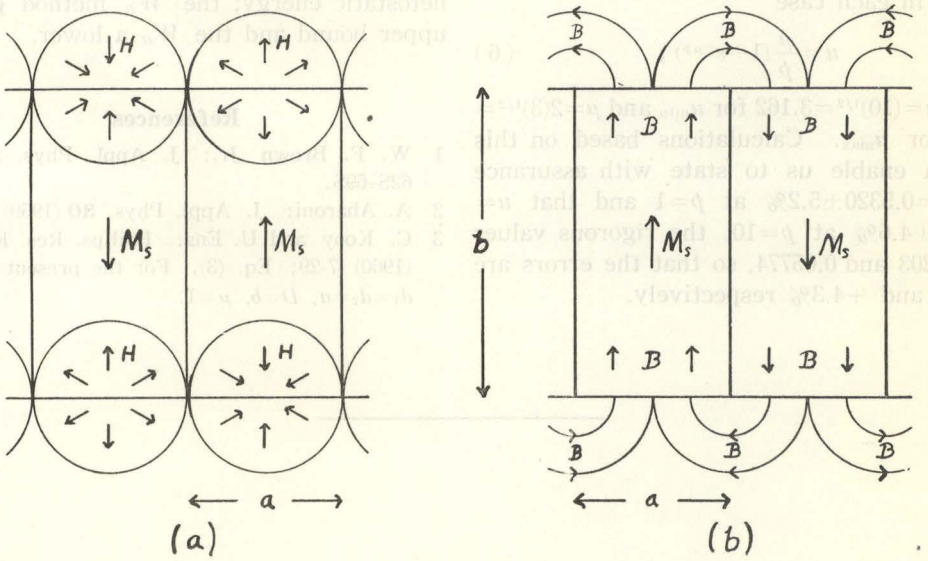

Fig. 1.

We may now compare this state with one of uniform horizontal magnetization, taking account of wall energy of surface density γ, and assuming that the uniform state is opposed by extra anisotropy energy of volume density K. Let $q=K/2\pi M_s^2$, $k=\pi M_s^2 b/\gamma$; the domain wall thickness is $\delta=\gamma/4\pi q M_s^2$. In general, the critical slab thickness b_c for domain formation is given by $k=-1/(du/dp)$, with p so chosen that $u-pdu/dp=2q$. If $q=0.3$, we get $(b_c)_{\min}=3.54\gamma/\pi M_s^2$, $(b_c)_{\max}=13.9\gamma/\pi M_s^2$; the corresponding domain widths, a, are $3.33\gamma/\pi M_s^2$ and $3.78\gamma/\pi M_s^2$ respectively, and $\delta=0.833\gamma/\pi M_s^2$. The calculation is much easier than the one based on Fourier series, and the accuracy is at least sufficient for preliminary estimates.

A more sophisticated way of treating the same problem is the following. To assure the irrotationality of $\boldsymbol{H}_2$ and the solenoidality of $\boldsymbol{B}_2$, derive them from a scalar potential ϕ and a vector potential $\Psi\boldsymbol{k}$ that are required to be continuous across all boundaries. Assume that ϕ and Ψ are each of the form $X(x)Y(y)$, where $X(x)$ is a conveniently chosen function, and determine $Y(y)$ so as to maximize W_H or minimize W_B. The resulting differential equation and boundary conditions are easily solved; the W's can then be expressed as simple functions of integrals that involve $X(x)$ and $X'(x)$. The following choices for $X(x)$ in the interval $(0, a)$ combine simplicity with the required symmetry: for ϕ, $x(a-x)$; for Ψ, $x-a/2$. Then we get in each case

$$u=\frac{\mu}{p}(1-e^{-\mu p}), \qquad (6)$$

where $\mu=(10)^{1/2}=3.162$ for $u_{\min}$ and $\mu=2(3)^{1/2}=3.464$ for $u_{\max}$. Calculations based on this formula enable us to state with assurance that $u=0.5320\pm5.2\%$ at $p=1$ and that $u=0.05523\pm4.6\%$ at $p=10$; the rigorous values are 0.5203 and 0.05774, so that the errors are $+2.2\%$ and $+4.3\%$ respectively.

If we now turn again to the problem of critical thickness for domain formation, we can simplify the algebra by replacing $e^{-\mu p}$ by $e^{-\mu}$ in $u_{\min}$ and by 0 in $u_{\max}$; the simplified formulas still give lower and upper limits if $p\geq1$. We thus find that b in units of $\gamma/\pi M_s^2$ is between 5.61 and 6.41, with a in the same units estimated as 3.33 in both cases. No greater precision would be significant in view of the approximations inherent in the model; and the formulas used in the numerical work were all elementary.

In micromagnetics[1),2)], the method can be used, for example, in estimating the nucleation field H_n for magnetization curling in a long crystal of square cross section (an iron whisker). The magnitude of the nucleation field is found in general by minimizing a certain quantity, one term in which is the magnetostatic energy. Minimization under added constraints gives an upper bound; minimization after omission of an essentially positive term gives a lower bound. We can obtain a preliminary upper bound by imposing the constraint that there be no poles; we can obtain a preliminary lower bound by methods that involve omission of the essentially positive magnetostatic energy. Both these devices evade the necessity for solving the magnetostatic potential problem even approximately. If the preliminary upper and lower bounds are not close enough together, we can get closer ones by using the present method of approximating the magnetostatic energy; the W_B method gives an upper bound and the W_H a lower.

References

1 W. F. Brown Jr.: J. Appl. Phys. **30** (1959) 62S-69S.
2 A. Aharoni: J. Appl. Phys. **30** (1959) 70S-78S.
3 C. Kooy and U. Enz: Philips. Res. Repts. **15** (1960) 7-29; Eq. (3). For the present case, set $d_1=d_2=a$, $D=b$, $\mu=1$.