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THE BOUNDARY BETWEEN THE SPIN-GLASS AND FERROMAGNETIC STATES OF THE BOND-RANDOM ISING MODEL

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The bond-random (±J) Ising model in the several lattices is considered in the cluster approximation. The boundary between the ferromagnetic and spin-glass states at T=0, p_{FG}, is obtained as the transition point from the asymmetric distribution of the effective fields to the symmetric distribution by solving the integral equation for the distribution function.

1. Introduction

We consider the spin-glass (SG) state in the bond-random (±J) Ising model. The statistical mechanics of the spin-glass in the bond-random Ising model is formulated in terms of the distribution function of the effective fields, and the integral equation of it is derived[1,2] in the pair (Bethe) approximation. The boundary between the paramagnetic (P) state and the spin-glass states, and that between the paramagnetic and ferromagnetic (F) states were obtained in the pair and cluster approximations for several lattices [3-5]. In the present paper, the integral equation for the distribution function of the effective fields in the cactus approximation is derived and solved exactly at T=0. The solutions include the P, SG, and F states. The bond concentrations of the transition point from the SG to F states is obtained as the point of junctions of the symmetric and asymmetric distributions.

2. Integral equation for the distribution function of the effective fields in the cactus approximation

We approximate the lattice by an appropriate cactus lattice composed of clusters (pair, triangle, square, tetrahedron, etc.), z_c of which ($ijk\ell\ldots, ij'k'\ell'\ldots, ij''k''\ell''\ldots, \ldots$) are connected at each vertex. The density matrices of the vertex $\rho^{(1)}(\sigma_i)$ and that of the cluster $ijk\ell\ldots$ are given by

$$\rho^{(1)}(\sigma_i) = \exp(\beta H_i^{(1)}\sigma_i) \tag{1}$$

$$\rho^{(c)}(\sigma_i,\sigma_j,\sigma_k,\sigma_\ell,\ldots) = \exp(\Sigma_{\mu\nu}\beta J_{\mu\nu}\sigma_\mu\sigma_\nu + \Sigma_\mu \beta H_\mu^{(c)}\sigma_\mu) \tag{2}$$

$H_i^{(1)}$ is the effective field at the site i contributed from all clusters connected at the site i. $H_i^{(c)}$ (which should be written as $H_{ijk\ell\ldots}^{(c)}$) is the effective field at the site i contributed from the outside of the cluster $ijk\ell\cdots$. $\Sigma_{\mu\nu}$ runs over all the connected bonds of the cluster $ijk\ell\cdots$, and Σ_μ runs over all the vertices of the cluster. We postulate the reducibility $\mathrm{tr}_{ijk\ell\ldots}\hat\rho^{(c)}_{ijk\ell\ldots} = \hat\rho_i^{(1)}$. ∧ denotes the normalization. The postulate holds rigorously for the cactus lattice considered here. $\mathrm{tr}_{jk\ldots}\rho^{(c)}(\sigma_i,\sigma_j,\sigma_k\ldots)$ is a linear function of σ_i and has a form

$$\Sigma_{\sigma_j\sigma_k\ldots}\rho^{(c)}(\sigma_i,\sigma_j,\sigma_k\ldots) = A_i \exp[(\beta H_i^{(c)} + \beta h_{ijk\ell\ldots})\sigma_i] \tag{3}$$

where

$$h_{ijk\ell\cdots} = (1/2\beta)\ell n[f(1)/f(-1)] \tag{4}$$

$$f(\sigma_i) = \Sigma_{\sigma_j\sigma_k\cdots} \exp(-\beta H_i^{(c)}\sigma_i)\rho^{(c)}(\sigma_i,\sigma_j,\sigma_k\cdots) \tag{5}$$

$h_{ijk\ell\cdots}$ is the effective field at the site i contributed from the inside of the cluster $ijk\ell\cdots$. Since

$$H_{ijk\ell\cdots}^{(1)} = h_{ijk\ell\cdots}+h_{ij'k'\ell'\cdots}+h_{ij''k''\ell''\cdots}+\cdots = H_{ijk\ell\cdots}^{(c)}+h_{ijk\ell\cdots} \tag{6}$$

we have recurrence relation for the distribution function $G_i(H_i^{(c)})$ of $H_{ijk\ell\cdots}^{(c)}$

$$G_i(H_i^{(c)}) = \int \delta(H_i^{(c)} - h_{ij'k'\ell'\cdots}-h_{ij''k''\ell''\cdots}-\cdots)$$
$$\times \Pi_{\mu\nu}P(J_{\mu\nu})dJ_{\mu\nu}\ \Pi'_\mu G_\mu(H_\mu^{(c)})dH_\mu^{(c)} \tag{7}$$

μ in Π'_μ runs for all vertices of clusters $ij'k'\ell'\ldots$, $ij''k''\ell''\ldots$, except i, and $\mu\nu$ in $\Pi_{\mu\nu}$ runs for all connected bonds in these clusters. In uniform states such as P, F, and SG states, we regard the distribution of the effective fields is independent of the site. Then the subscript of G is dropped. Since there Π can be grouped in separate clusters, Fourier transform can be used and we have an integral equation.

$$G(H_i^{(c)}) = \frac{1}{2\pi}\int dk\ \exp(ikH_i^{(c)})[S(k)]^{z_c} \tag{8}$$

$$S(k) = \int \exp(-ikh_{ij'k'\ell'\cdots})\ \Pi_{\mu\nu}P(J_{\mu\nu})dJ_{\mu\nu}\Pi'_\mu G(H_\mu^{(c)})dH_\mu^{(c)} \tag{9}$$

μ in Π'_μ runs on the vertices of the cluster $ij'k'\ell'\ldots$ except i, and $\mu\nu$ in $\Pi_{\mu\nu}$ runs on the connected bonds in the cluster.

We assume that the distribution of the single cluster effective fields as superposition of delta functions, i.e.,

$$S(k) = \Sigma_n\mu_n\exp(-inJk) \tag{10}$$

Substituting (11) into (9) we have

$$G(H_i^{(c)}) = \Sigma_m a_m\delta(H_i^{(c)}-mJ) \tag{11}$$

in the limit $T\to 0$. Inserting (11) into (9) we have new S(k) as

$$S(k) = \Sigma_n\mu'_n\exp(-inJk) \tag{12}$$

μ'_n is expressed in terms of a_m and a_m in terms of μ_n. Hence we have algebraic equations of μ_n. The values which satisfy these relation give the solution of the integral equation. The solutions are classified in the following types. Single peaked δ-function at $H_i^{(c)} = 0$ describes the P state. Asymmetric solution describes the F state. Symmetric solution with width describes the SG state. The value of the concentration of the ferromagnetic bond where the asymmetric solution begins to appear, and connects to the symmetric solution, p_{FG}, is the phase boundary between the F and SG states.

3. Square lattice in the square cluster approximation

We consider a square cactus lattice of $z_c=2$ as an approximation of the square lattice. For $z_c=2$, μ_n is equal to a_n. Substitution of

$$P(J_{\mu\nu}) = p\delta(J_{\mu\nu}-J)+(1-p)\delta(J_{\mu\nu}+J)$$

and

$$S(k) = \Sigma_{\ell=-2}^{2} a_\ell \exp(-i\ell Jk)$$

(p is the concentration of the ferromagnetic bonds) into the integral equation (9) and (10) we have

$$\Sigma_{\ell} a_{\ell} \exp(-inJk) = \Sigma_{mnqPQRS}\, a_m a_n a_q\, p^{(4+P+Q+R+S)/2} (1-p)^{(4-P-Q-R-S)/2}$$

$$\times \exp[-i(u_{mnqPQRS}(1) - u_{mnqPQRS}(-1))kJ/2] \qquad (13)$$

where the current indices m,n,q run from -2 to 2, and those of P,Q,R,S run 1 and -1, and

$$u_{mnqPQRS}(\sigma_i) = \max_{\sigma_j \sigma_k \sigma_\ell} (P\sigma_i\sigma_j + Q\sigma_j\sigma_k + R\sigma_k\sigma_\ell + S\sigma_\ell\sigma_i + m\sigma_j + n\sigma_\ell + q\sigma_k) \qquad (14)$$

From the same powers of rhs and lhs of (14) we have 5 equations. Introducing 4 variables

$$z = (a_{-2}+a_2)/2,\ y = (a_{-1}+a_1)/2,\ v = (a_{-2}-a_2)/2,\ w = (a_{-1}-a_1)/2 \qquad (15)$$

instead of a_n (v and w measure the asymmetry of the distribution), we have 4 equations. They are solved as a function of $x \equiv p-1/2$, and give relevant solutions:1) P state. $a_0=1$ in the whole region of x. 2) SG1 state. v=0, w=0, y=0. 3) SG2 state. v=0, w=0, $y\neq 0$. 4) F1 state. $v\neq 0$, w=0, y=0, $0.335 < x < 0.5$. 5) F2 state. $v\neq 0$, $w\neq 0$, $y\neq 0$, $0.327 < x < 0.375$. The value of z,y,v,w are shown in Fig. 1. SG1 and F1 connect at $p_{FG1}=0.835$. SG2 and F2 connect at $p_{FG2}=0.827$. Such features are similar as in the pair approximation [6]. Details will be published in [7].

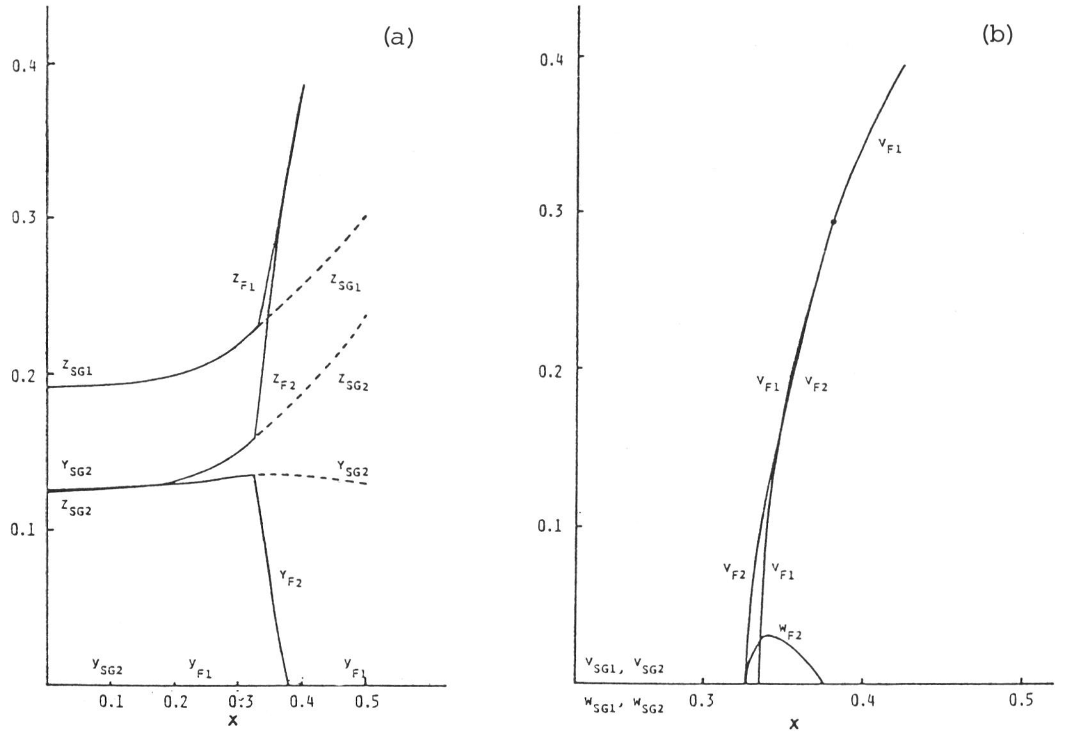

Fig.1 The distribution function of the effective fields for the square lattice in the square cactus approximation as a function of the concentration. $x \equiv p-1/2$. $G(H^{(4)}) = \Sigma_{m=-2}^{2} a_m \delta(H^{(4)}-mJ)$. The ordinate, z,y,v,w are given in (15).

In a similar way we solved the integral equations for the distribution functions of the effective fields in several lattices at T=0 and calculated the values p_{FG}. The results are shown in Table 1. These values of p_{FG} at T=0 together with the values of the tricritical point p_t (cross point of the P-F and P-SG boundaries) already obtained [3-5] help us to estimate the phase boundaries for finite temperatures. The discussions on SG1 and SG2 are given in [6,8].

Table 1 Boundary between the ferromagnetic and spin glass states p_{FG} and p_t. ( ) denotes the number of δ-functions in the distribution of single cluster field at T=0.

Lattice	Approximation	p_{FG1}	p_{FG2}	p_t
Hexagonal	pair z=3	0.8750 (3)		0.85355
Square	pair z=4	0.8333 (2)	0.8205 (3)	0.78869
	sq cactus z_c=2	0.835 (3)	0.827 (5)	0.80145
Kagome	tr cactus z_c=2	0.854 (3)	0.850 (5)	0.82949
Triangular	tr cactus z_c=3	0.766 (3)	0.761 (5)	0.7346

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