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Two Phase Transitions in Ashkin-Teller Model

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Critical temperature of Ashkin-Teller model is calculated in the molecular field approximation. Generally it has two critical temperatures when all the positive coupling constants are different. When one of the coupling constants becomes negative, the lower critical temperature decreases and disappears by competition between the positive and the negative couplings of two layers.

1. Introduction

Originally the Ashkin-Teller model (AT model) which is a model of mixture was introduced as an extension of the Ising model [1]. Equivalence to two-layered Ising model with four-spin interaction was proved by Fan [2]. Furthermore AT model was shown to be important after the proof of equivalence to a staggered eight vertex model [3]. Wu and Lin conjectured the existence of two phase transitions in AT model by use of the above equivalence and several limiting cases whose exact solutions are known when all the coupling constants are different [4].

This paper is also concerned with two phase transitions. The two critical temperatures are calculated explicitly within the molecular field approximation. Especially when one of the coupling constants becomes negative, one of critical temperatures decreases and becomes zero by competition between the positive and the negative coupling constants. It can be expected that a new phase occurs for a system with such parameters.

2. Formulation

The Hamiltonian for AT model is expressed by

$$H = J_1 \sum_{<ij>} \sigma_i\sigma_j + J_2 \sum_{<i'j'>} \tau_{i'}\tau_{j'} + J_3 \sum_{<ij>} \sum_{<i'j'>} \sigma_i\sigma_j\tau_{i'}\tau_{j'} \quad , \tag{1}$$

in the Ising spin language, where σ and τ are the Ising spin variables for each layer. The first and the second summations are taken over all the nearest neighbor pairs and the third one all the plaquette lattices between the two layers.

Let us introduce the expectation such as

$$\lambda_1 = <\sigma> \ , \ \lambda_2 = <\tau> \ \text{and} \ \ \lambda_3 = <\sigma\tau> \quad . \tag{2}$$

λ_1, λ_2 and λ_3 are determined from the following equations:

$$\begin{aligned} \lambda_1 &= \{f(1,1,1)+f(1,-1,-1)-f(-1,1,-1)-f(-1,-1,1)\}/Z \quad , \\ \lambda_2 &= \{f(1,1,1)-f(1,-1,-1)+f(-1,1,-1)-f(-1,-1,1)\}/Z \quad , \\ \lambda_3 &= \{f(1,1,1)-f(1,-1,-1)-f(-1,1,-1)+f(-1,-1,1)\}/Z \end{aligned} \tag{3}$$

and

$$Z = f(1,1,1)+f(1,-1,-1)+f(-1,1,-1)+f(-1,-1,1) \quad , \tag{4}$$

where

$$f(a,b,c) = \exp(aK_1 z\lambda_1+bK_2 z\lambda_2+cK_3 z\lambda_3) \quad , \tag{5}$$

$$K_i = J_i/kT$$

and z is the coordination number.

When one of three λ's is zero, the above equations are reduced to

$$\lambda_{i(j)} = \tanh K_{i(j)} z\lambda_{i(j)} \quad \text{for } \lambda_k = 0 \quad , \tag{6}$$

where i,j and k are one of the permutations of 1,2 and 3. Therefore the critical behavior is the same as the usual 2-dimensional Ising model, that is,

$$kT_c^{i(j)} = zJ_{i(j)} \quad \text{for } \lambda_k = 0 \quad .$$

When one coupling constant becomes negative, a two-sublattice spin structure should be taken into account as follows:

$$\begin{aligned} \lambda_1^A &= \sum_{\{\sigma,\tau\}} \sigma^A \exp(\tilde{H}_{eff})/Z' \quad , \\ \lambda_2^A &= \sum_{\{\sigma,\tau\}} \sigma^A \exp(\tilde{H}_{eff})/Z' \quad , \\ \lambda_3 &= \sum_{\{\sigma,\tau\}} \sigma^A \tau^A \exp(\tilde{H}_{eff})/Z' \end{aligned} \tag{7}$$

and the formulae for λ_1^B, λ_2^B and λ_3^B are obtained by replacing A with B in eq.(7) where

$$\tilde{H}_{eff} = K_1(\sigma^A\lambda_1^B+\sigma^B\lambda_1^A)+K_2(\tau^A\lambda_2^B+\tau^B\lambda_2^A)+K_3(\sigma^A\tau^A\lambda_3^B+\sigma^B\tau^B\lambda_3^A) \tag{8}$$

and

$$Z' = \sum_{\{\sigma,\tau\}} \exp(\tilde{H}_{eff}) \quad . \tag{9}$$

The superscript A and B are introduced to distinguish the A and B sublattices.

3. Results

In a general case with all non-zero J's we calculated the set of eqs.(3) by a numerical way. The result is shown in Fig.1(a) and Fig.1(b). Fig.1(a) shows the critical temperature versus J_3 for the case of J_1=2 and J_2=1. For relatively small and positive values of J_3 we have two phase transition points T_{c1} and T_{c2} ($T_{c1}<T_{c2}$). Two order parameters have non-zero values below T_{c1} and λ_3 is also non-zero. For $T_{c1}<T<T_{c2}$, only λ_1 exists. For the case of large J_3, to the contrary, we have a similar state as the previous case except the fact that only λ_3 is non-zero for $T_{c1}<T<T_{c2}$ even though T_{c1} and T_{c2} are different from the previous case.

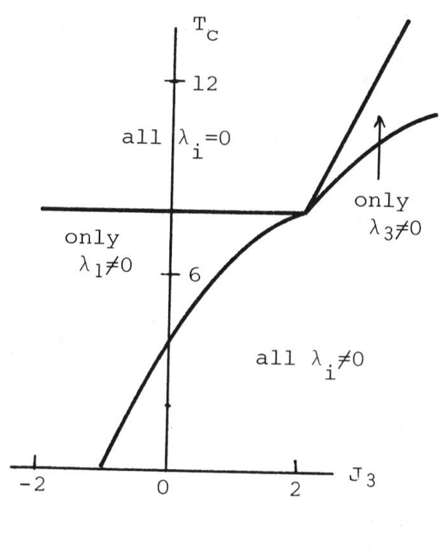

Fig.1(a) The critical temperature versus J_3 when J_1=2 and J_2=1.

Figure 1(b) shows the critical temperature versus J_3 for the case of $J_1= J_2=1$. When J_3 is relatively large, the behavior of critical temperature is similar to the case (a). However when J_3 is small, we have always only one critical temperature.

Now we wish to discuss the case of J_3 negative. T_{c2} keeps the same value as the critical temperature at $J_3=0$. T_{c2}, however, decreases as the absolute value of J_3 increased. This phenomenon is due to the competition between the positive and the negative coupling constants. We have a possibility of new phase in the system with such parameters as T_{c1} vanishes, although it is not known yet.

In the case (b) such a character as only one critical temperature is maintained even for J_3 negative like positive side of J_3. We cannot expect a new phase in this case, but we have a ferromagnetically ordered state even with energy loss caused by the negative coupling constant between two layers. Figure 2 shows the magnetization for two cases, (i) $J_1=2$, $J_2=1$, $J_3=0.5$ and (ii) $J_1=2$, $J_2=1$, $J_3=3.5$ of (a).

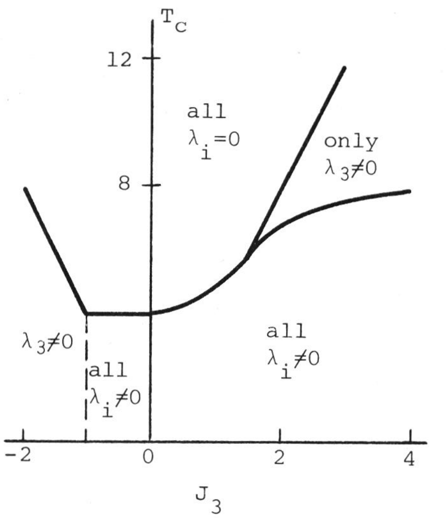

Fig.1(b) The critical temperature versus J_3 when $J_1=J_2=1$.

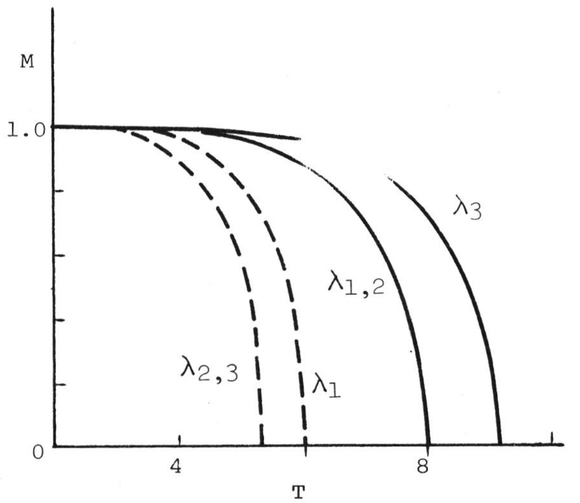

Fig.2 The magnetization for (i) $J_1=2$, $J_2=1$, $J_3=0.5$ (dashed curve) and (ii) $J_1=2$, $J_2=1$, $J_3=3.5$ (solid curve).

4. Discussion

The four-spin interaction gives us an impression that it is a very weak interaction at first, because AT model has two critical temperatures like two layers are nearly independent. The higher critical temperature T_{c2} is equal to J_1z in the molecular field approximation, although the lower one is modified from J_2z by the four-spin interaction. On the other hand it is easily understood by applying a transformation $\sigma \to \sigma'\tau$ or $\tau'\sigma$ that three terms in (1) exchange their roll each other.

If we consider a two-layered Ising lattice with two-spin interaction between two layers instead of four-spin interaction, we have only one critical temperature no matter how much three parameters are. It is an interesting problem that we consider two-layered Ising model with usual two-spin interaction, but one layer has ferromagnetic interaction and the other antiferromagnetic.

References

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